# Bayesian Pose Graph Optimization via Bingham Distributions and Tempered Geodesic MCMC

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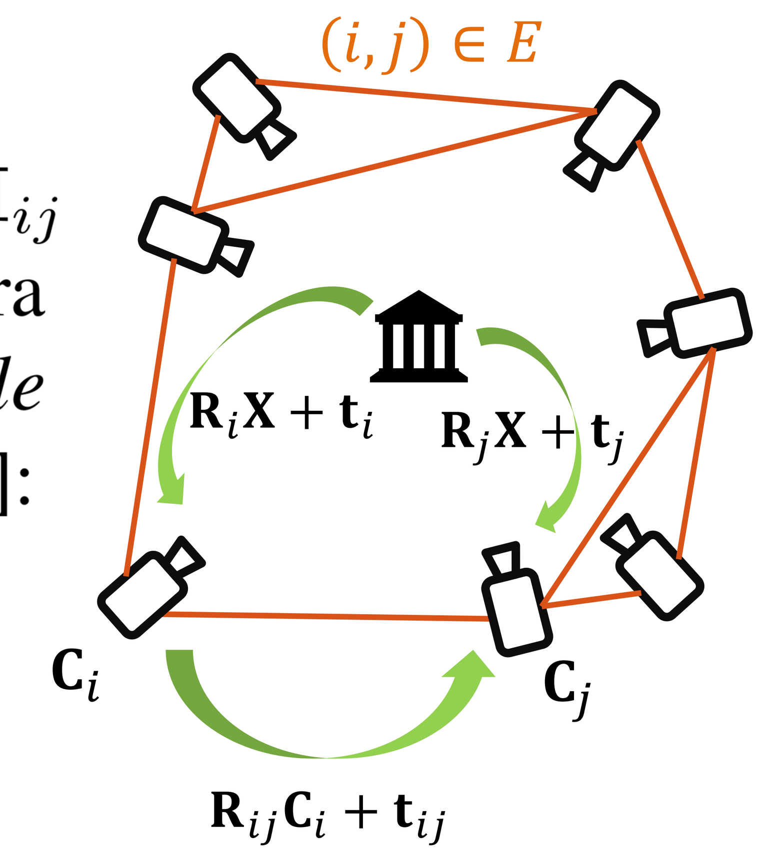
## Introduction

### Multiple Motion Averaging

Given the *relative* pose estimates $\{\mathbf{M}_{ij} \in SE(3)\}$, find the *absolute* camera poses $\{\mathbf{M}_i \in SE(3)\}$ s.t. the *cycle consistency constraint* is satisfied [1]:

$$\mathbf{M}_{ij} \approx \mathbf{M}_j\mathbf{M}_i^{-1},\ \forall i \neq j$$




### Contributions

1. With observations $\mathcal{D} \equiv \{\mathbf{q}_{ij}, \mathbf{t}_{ij}\}_{(i,j)\in E}$ and latent variables (unknown) $\mathbf{Q} \equiv \{\mathbf{q}_i\}_{i=1}^n$, $\mathbf{T} \equiv \{\mathbf{t}_i\}_{i=1}^n$ and $\mathbf{x} = \{\mathbf{Q}, \mathbf{T}\}$, compute MAP estimate:

$$(\mathbf{Q}^\star, \mathbf{T}^\star) = \arg\max_{\mathbf{Q},\mathbf{T}} p(\mathbf{Q},\mathbf{T}|\mathcal{D}) =$$

$$\arg\max_{\mathbf{Q},\mathbf{T}} \Big( \sum_{(i,j)\in E} \{\log p(\mathbf{q}_{ij}|\mathbf{Q},\mathbf{T}) + \log p(\mathbf{t}_{ij}|\mathbf{Q},\mathbf{T})\} + \sum_i \log p(\mathbf{q}_i) + \sum_i \log p(\mathbf{t}_i)\Big)$$

Novel probabilistic model

2. Obtain the full posterior distribution (via MCMC):

$$p(\mathbf{Q},\mathbf{T}|\mathcal{D}) \propto p(\mathcal{D}|\mathbf{Q},\mathbf{T}) \times p(\mathbf{Q}) \times p(\mathbf{T})$$

3. with theoretical guarantees:

$$|\mathbb{E}\hat{U}_N - U^\star| = \mathcal{O}\Big(\frac{\beta}{Nh} + \frac{h}{\beta} + \frac{1}{\beta}\Big)$$

Samples close to the global minimum

4. controlled by a single inverse temperature parameter $\beta$.

$\beta\ =\ 1$ : SG-GMC [2]
$\beta\ \to \infty$ : Riemannian-GD + Momentum

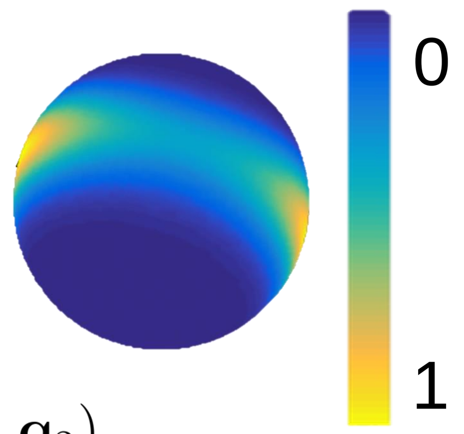
$\mathbf{q}$ : quaternion
$\mathbf{t}$ : translation
$U(\mathbf{x}) = -\log p(\mathcal{D},\mathbf{x})$
$\mathbf{x}^\star$ : global optimum
$U^\star \triangleq U(\mathbf{x}^\star)$
$\hat{U}_N$ : sample average
$N$ : iterations
$h$ : step-size

## Quaternions & Bingham Distributions

An antipodally symmetric probability distribution lying on $\mathbb{S}^{d-1}$ with PDF $\mathcal{B}: \mathbb{S}^{d-1} \to R$:

$$\mathcal{B}(\mathbf{x};\mathbf{\Lambda},\mathbf{V}) = \frac{1}{F}\exp(\mathbf{x}^T\mathbf{V}\mathbf{\Lambda}\mathbf{V}^T\mathbf{x}) = \frac{1}{F}\exp\Big(\sum_{i=1}^d \lambda_i(\mathbf{v}_i^T\mathbf{x})^2\Big)$$

$$d_{\text{riemann}} = \|\log(\mathbf{R}_1\mathbf{R}_2^T)\| = 2\arccos(|\mathbf{q}_1\bar{\mathbf{q}}_2|) = d_{\text{bingham}}(\mathbf{q}_1,\mathbf{q}_2)$$




## Proposed Model

$$\mathbf{q}_i \sim p(\mathbf{q}_i), \quad \mathbf{t}_i \sim p(\mathbf{t}_i), \quad \mathbf{q}_{ij}|\cdot \sim p(\mathbf{q}_{ij}|\mathbf{q}_i,\mathbf{q}_j), \quad \mathbf{t}_{ij}|\cdot \sim p(\mathbf{t}_{ij}|\mathbf{q}_i,\mathbf{q}_j,\mathbf{t}_i,\mathbf{t}_j)$$

$$\mathbf{q}_{ij}|\mathbf{q}_i,\mathbf{q}_j \sim \mathcal{B}(\mathbf{\Lambda},\mathbf{V}(\mathbf{q}_j\bar{\mathbf{q}}_i)), \qquad \mathbf{t}_{ij}|\mathbf{q}_i,\mathbf{q}_j,\mathbf{t}_i,\mathbf{t}_j \sim \mathcal{N}(\boldsymbol{\mu}_{ij},\sigma^2\mathbf{I})$$

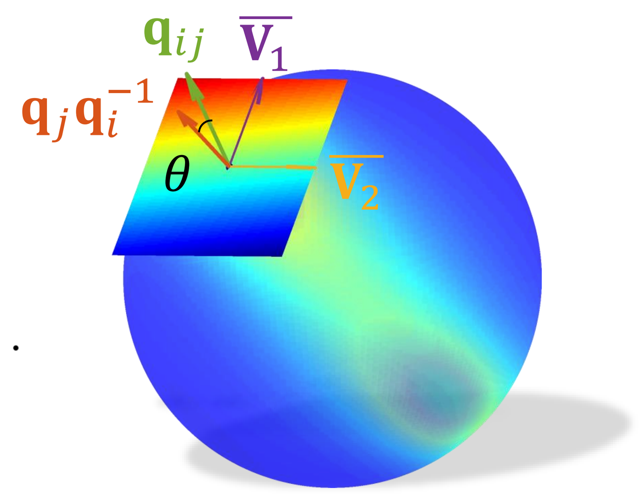
mode ≡ most likely relative pose:

$$\arg\max_{\mathbf{q}_{ij}} \{p(\mathbf{q}_{ij}|\mathbf{q}_i,\mathbf{q}_j) = \mathcal{B}(\mathbf{\Lambda},\mathbf{V}(\mathbf{q}_j\bar{\mathbf{q}}_i))\} = \mathbf{q}_j\bar{\mathbf{q}}_i.$$

$S^3$ is a parallelizable manifold:

$$\mathbf{V}(\mathbf{q}) \triangleq \begin{bmatrix} q_1 & -q_2 & -q_3 & q_4 \\ q_2 & q_1 & q_4 & q_3 \\ q_3 & -q_4 & q_1 & -q_2 \\ q_4 & q_3 & -q_2 & -q_1 \end{bmatrix}.$$

$$\boldsymbol{\mu}_{ij} \triangleq \mathbf{t}_j - (\mathbf{q}_j\bar{\mathbf{q}}_i)\mathbf{t}_i(\mathbf{q}_i\bar{\mathbf{q}}_j)$$




## Inference: Tempered Geodesic MCMC (TG-MCMC)

### Proposed SDE

$$d\tilde{\mathbf{x}}_t = \mathbf{G}(\tilde{\mathbf{x}}_t)^{-1}\mathbf{p}_t dt$$

$$d\mathbf{p}_t = -\Big(\nabla_{\tilde{\mathbf{x}}}U_\lambda(\tilde{\mathbf{x}}_t) + \frac{1}{2}\nabla_{\tilde{\mathbf{x}}}\log|\mathbf{G}| + c\mathbf{p}_t + \frac{1}{2}\nabla_{\tilde{\mathbf{x}}}(\mathbf{p}_t^\top\mathbf{G}^{-1}\mathbf{p}_t)\Big)dt + \sqrt{(2c/\beta)\mathbf{M}^\top\mathbf{M}}\, dW_t$$

### Split SDE

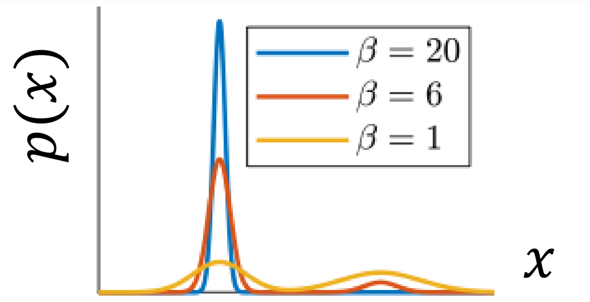
$$\text{A:}\begin{cases} d\tilde{\mathbf{x}}_t = \mathbf{G}^{-1}\mathbf{p}_t dt \\ d\mathbf{p}_t = \frac{1}{2}\nabla(\mathbf{p}_t^\top\mathbf{G}^{-1}\mathbf{p}_t)dt \end{cases}$$

$$\text{B:}\begin{cases} d\tilde{\mathbf{x}}_t = 0 \\ d\mathbf{p}_t = -c\mathbf{p}_t dt \end{cases}$$

$$\text{O:}\begin{cases} d\tilde{\mathbf{x}}_t = 0 \\ d\mathbf{p}_t = -(\nabla U_\lambda(\tilde{\mathbf{x}}_t) + \frac{1}{2}\nabla\log|\mathbf{G}|)dt + \sqrt{\frac{2c}{\beta}\mathbf{M}^\top\mathbf{M}}dW_t. \end{cases}$$


Hamiltonian Paths converge to a measure with density: $e^{-\beta U(\mathbf{x})}$

$\mathbf{G}$ : Riemannian metric, $c$ : friction, $\mathbf{M} \triangleq \partial\mathbf{x}_i/\partial\tilde{\mathbf{x}}_j$
$\mathbf{p}$ : momentum, $dW_t$ : Brownian motion

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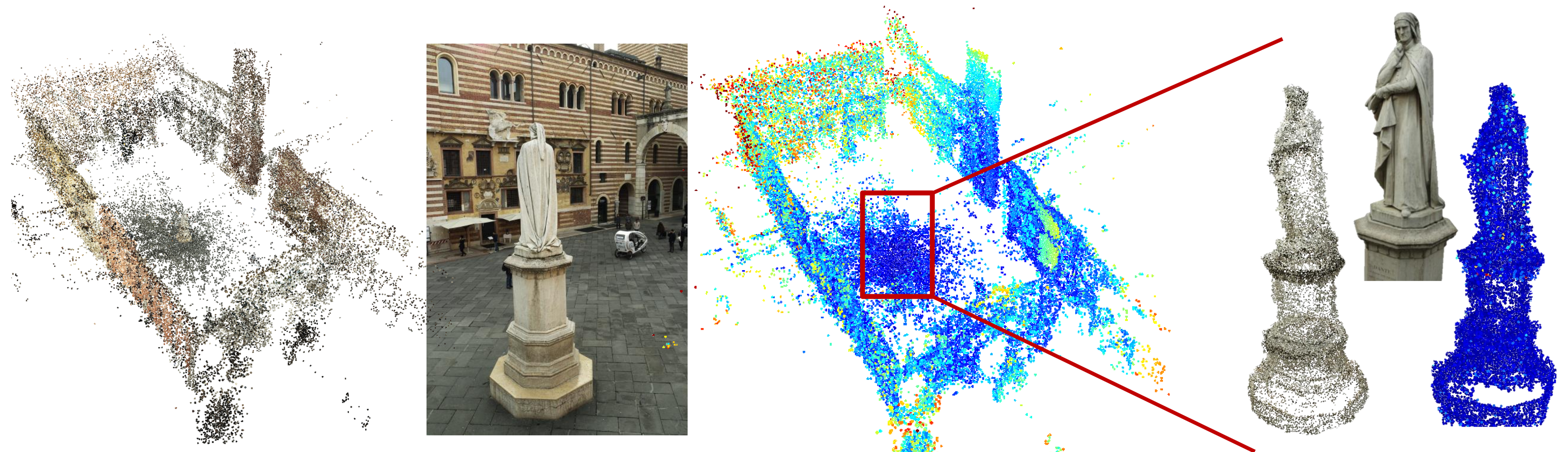
input: x_0 = {q_1, ..., q_n, t_1, ..., t_n}, v = {v_1^q, ..., v_n^q, v_1^t, ..., v_n^t}, β, c, h
for i = 1, ..., N do
    Compute the gradient ∇_x U(x_i)
    // Update the rotation components
    for j = 1, ..., n do
        Run the B, O, A steps (in this order) on q_j, v_j^q
    // Update the translation components
    for j = 1, ..., n do
        Run the B, O, A steps (in this order) on t_j, v_j^t
```

Results in a simple algorithm

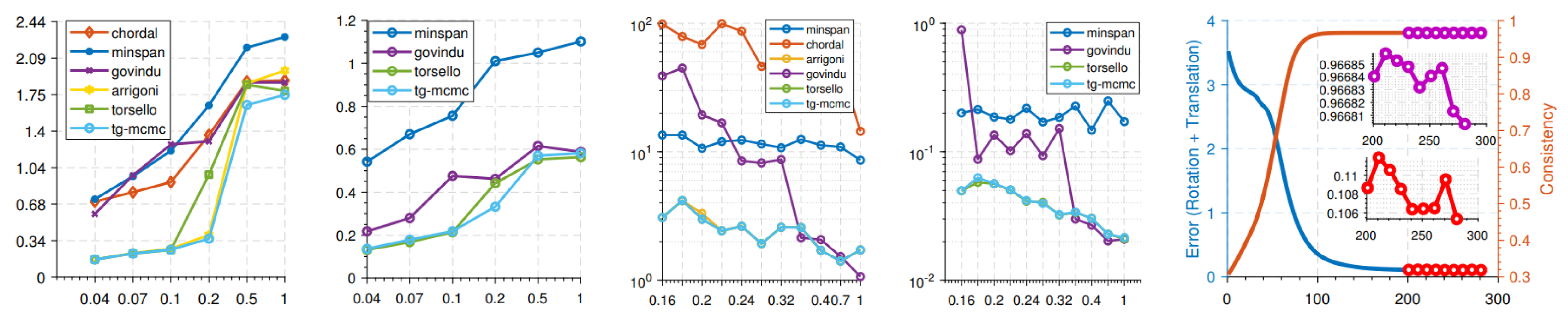
## Evaluations

### Qualitative

**(a)** Madrid Metropolis   **(b)** 3D Reconstruction   **(c)** Uncertainty Map



### Quantitative



rot-noise   trans-noise   rot-|E|%   trans-|E|%   opt-sample

### References

1. Govindu, Venu Madhav. "Combining two-view constraints for motion estimation." *CVPR 2001.*
2. Liu, Chang, Jun Zhu, and Yang Song. "Stochastic gradient geodesic MCMC methods." *NIPS* 2016.
3. Arrigoni, Federica, Andrea Fusiello, and Beatrice Rossi. "Camera motion from group synchronization." *3DV 2016.*
4. Torsello, A., Emanuele R., and Andrea A.. "Multiview registration via graph diffusion of dual quaternions." *CVPR 2011*.